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Binary bases of spaces of continuous functions

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Abstract

This paper addresses the question of density of binomials in $C[a,b]$. This seemingly innocuous question turns out to be equivalent in certain instances to the moment problem.

1. Introduction

In this paper we consider the following problem. Let $\alpha = (\alpha_0, \alpha_1, \ldots)$ and set

$$S_\alpha = \operatorname{span}\{t^n - \alpha_n t^{n+1}\colon n = 0, 1, \ldots\},$$

and let

$$C[a,b]$$

be the Banach space of continuous functions on the finite interval $[a,b]$ with norm defined by

$$\|f\| = \max_{t\in[a,b]} \left|f(t)\right|.$$

When is S_α dense in $C[a,b]$? We will show that this problem is quite complex and, in fact, variants of this problem are equivalent to the general moment problem.

This problem arises as a special case of a problem in the approximation of continuous functions considered in [3]. Given a linear control system of the form

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$$\dot{x} = Ax + bu,$$
$$y = cx,$$

where $x \in \mathbb{R}^n$, $y, u \in \mathbb{R}$ and A, b and c are real constant matrices of compatible dimension a problem of some interest is to determine what functions can be approximated by $y(t)$ for a choice of controls u. See, for example, the papers [6–9] and references therein. It is a straightforward problem to determine just what functions can be approximated when the control is taken to be an arbitrary continuous function. In [3] an effort was made to determine rates of convergence of y when the input was restricted to polynomials. While doing that the question arose of rates of approximation when simple classes of polynomials are used. It was realized that it was not known just how well $C[a, b]$ is approximated in many cases. The simplest case seems to be to ask if the span of sets of binomials approximates every continuous function. If so then we can use sums of binomials instead of arbitrary polynomials. However when this problem was pursued it was seen to be difficult to completely answer and the pursuit of the problem led to the work presented in this paper.

In Section 2 we will establish some basic facts about S_α and its closure that are of some interest in themselves and will be used in the rest of the paper. In Section 3 we will show that the problem is related to a classical moment problem. Section 4 relates the general moment problem to the question of density of a certain set of polynomials that is related to (and sometimes equal to) S_α. In Section 5 we show that the corresponding problem for Hilbert space is somewhat simpler and provides a partial answer to the question of density in $C[a, b]$. In particular we treat both $L^2[a, b]$ and the Hardy space H^2. In Section 6 we show that the problem is related to a series of operator questions and we explore the relation of these operator equations to the moment problem.

2. Basics

In this section we establish some basic facts and notation and prove the first sufficient condition for S_α to be dense $C[a, b]$. We will use the notation $[f_\alpha\colon \alpha \in A]$ to denote the closed linear span of the set $\{f_\alpha\colon \alpha \in A\}$ with respect to $C[a, b]$. The notation $[f_\alpha\colon \alpha \in A]_X$ will denote the closed linear span of the set $\{f_\alpha\colon \alpha \in A\}$ with respect to the Banach space X. For example, it is well known that $[t^k\colon k = 0, 1, \ldots] = C[a, b]$.

We begin with the following lemma, which is proved by simple manipulation.

Lemma 2.1.

$$t^m - \left(\prod_{k=m}^{n} \alpha_k\right) t^{n+1} \in S_\alpha.$$

The next several corollaries establish results that are of interest, but the most important use will be to allow us to remove a lot of unnecessary detail. Their proofs are a direct consequence of Lemma 2.1.

We first point out what happens when one of the α_n's vanishes.

Corollary 2.2. *If $\alpha_n = 0$ then for every $k \leqslant n$, $t^k \in S_\alpha$.*

Now consider the case for an infinite number of α_n's vanishing.

Corollary 2.3. *If the set* $\{n\colon\ \alpha_n = 0\}$ *is infinite then*

$$[t^k - \alpha_k t^{k+1}\colon\ k \geqslant 0] = C[a,b].$$

Finally, if we have control on the growth of the products of the α_n's then we can state the following corollary.

Corollary 2.4. *Suppose* $a < b$ *and*

$$\lim_{n\to\infty}\left(\prod_{k=0}^{n}\alpha_k\right)a^{n+1} = 0, \qquad \lim_{n\to\infty}\left(\prod_{k=0}^{n}\alpha_k\right)b^{n+1} = 0.$$

Then for every $j = 0, 1, \ldots$

$$t^j \in [t^k - \alpha_k t^{k+1}\colon\ k \geqslant 0] = C[a,b].$$

We now prove a simple but important result.

Proposition 2.5. *Let* $\alpha_{n_0} = 0$ *and assume that for* $n > n_0$ $\alpha_n \neq 0$*. If*

$$t^{n_0+1} \in [t^k - \alpha_k t^{k+1}\colon\ k \geqslant 0]$$

then

$$[t^k - \alpha_k t^{k+1}\colon\ k \geqslant 0] = C[a,b].$$

Proof. The proof of the proposition is established by recursively showing that $t^m \in [t^k - \alpha_k t^{k+1}\colon\ k \geqslant 0]$ for every $m > n_0$ and appealing to the Weierstrass approximation theorem to conclude that $[t^k - \alpha_k t^{k+1}\colon\ k \geqslant 0] = C[a,b]$. $\square$

We now establish the first result that is of some general importance.

Theorem 2.6. *For all* k *let* $\alpha_k = c$*. Then* $[t^k - \alpha_k t^{k+1}\colon\ k \geqslant 0] = C[a,b]$ *if and only if* $1/c \notin [a,b]$.

Proof. It is easy to see that for any choice of $(a_0, \ldots, a_n)$

$$a_0 + a_1 t + \cdots + a_n t^n - c(a_0 t + a_1 t^2 + \cdots + a_n t^{n+1}) \in S_\alpha.$$

Thus we have by just rewriting that $p(t)(1 - ct) \in S_\alpha$ for every polynomial $p(t)$. Again using the Weierstrass approximation theorem we have that for every continuous function $f(t)$, $f(t)(1 - ct) \in [t^k - \alpha_k t^{k+1}\colon\ k \geqslant 0]$. Now $1/(1 - ct) \in C[a,b]$ if and only if $1/c \notin [a,b]$ and hence we have that $1 \in [t^k - \alpha_k t^{k+1}\colon\ k \geqslant 0]$ if and only if $1/c \notin [a,b]$. Using Proposition 2.5 the theorem follows. $\square$

3. Relation to the general moment problem

The primary tool that we will use in this section is the separating hyper-plane theorem. We also use various facts about the Banach space $C[a,b]$. The version of the separating hyper-plane theorem which we will use in this paper is based on theorems in Luenberger [4]. We state the following without proof and refer the reader to [4].

Theorem 3.1 (separating hyper-plane theorem). *Let B be a Banach space and let V be a closed linear subspace of B. Let $a \in B$ be a point in B not in V. Then there exists a linear functional F on B such that for each $v \in V$, $F(v) = 0$ and $F(a) = 1$.*

A complete characterization of the continuous linear functionals on $C[a,b]$ is known, see for example [5]. We state the following theorem, again without proof, and refer the reader to [5].

Theorem 3.2 (the dual of $C[a,b]$). *Let $v(t)$ be a function of bounded variation on the interval $[a,b]$. Then the linear functional*

$$F(f) = \int_a^b f(t)\,dv(t)$$

is a continuous linear functional defined on $C[a,b]$. Let F be a continuous linear functional defined on $C[a,b]$. There exists a function of bounded variation defined on the interval $[a,b]$ such that for every $f \in C[a,b]$, $F(f) = \int_a^b f(t)\,dv(t)$.

To apply these theorems we let

$$V = [t^k - \alpha_k t^{k+1}\colon\ k \geqslant 0].$$

We assume for the time being that none of the α_ks are equal to zero. Then it follows that $1 \in V$ if and only if $V = C[a,b]$. So we assume that 1 is not in V. Thus by the separating hyper-plane theorem there exists a continuous linear functional F so that

$$F(t^k - \alpha_k t^{k+1}) = 0$$

for all k and

$$F(1) = 1.$$

We can now state the following theorem whose proof follows from the preceding discussion.

Theorem 3.3. $[t^k - \alpha_k t^{k+1}\colon\ k \geqslant 0] = C[a,b]$ *if and only if the unique solution of the infinite set of equations*

$$\beta_0 = 1,$$
$$\beta_0 - \alpha_0\beta_1 = 0,$$
$$\beta_1 - \alpha_1\beta_2 = 0,$$
$$\vdots$$

is such that the linear functional defined by

$$F(t^k) = \beta_k$$

has no continuous extension to $C[a,b]$.

Proof. Suppose that $[t^k - \alpha_k t^{k+1}\colon k \geqslant 0] = C[a,b]$. Then any continuous linear functional that vanishes on S_α must vanish identically. The F determined by the linear relations must vanish on S_α and hence F must be zero if it is to be continuous. But this means that $F(1) = 0$ which is in conflict with $F(t^0) = 1$. This contradiction proves the forward implication. Conversely if there is no continuous extension then there is no continuous F that annihilates S_α and satisfies $F(1) = 1$. Thus $t^0 \in [t^k - \alpha_k t^{k+1}\colon k \geqslant 0]$ and hence by Proposition 2.5 $[t^k - \alpha_k t^{k+1}\colon k \geqslant 0] = C[a,b]$. □

Examples

Example 1. $\{\alpha_k = c\}$. Consider the subspace $[t^k - ct^{k+1}\colon k \geqslant 0]$. As discussed earlier we know that this subspace is dense in $C[a,b]$ if and only if $1/c \notin [a,b]$. If $z = 1/c \in [a,b]$ then the delta functional $\delta_z \in C[a,b]^*$ and $\delta_z \perp [t^k - ct^{k+1}\colon k \geqslant 0]$ confirming via the last theorem that this space is not all of $C[a,b]$. If $z = 1/c \notin [a,b]$ then the functional δ_z defined by $p \to p(z)$ is well defined on S_α and it annihilates S_α. However, it is easy to see that there is no continuous extension to $C[a,b]$.

Example 2. $\{\alpha_k = (k+2)/(k+1)\}$ on $C[0,2]$. The subspace $[t^k - (k+2)/(k+1)t^{k+1}\colon k \geqslant 0]$ is not equal to $C[0,2]$ since the continuous linear functional (on $C[0,2]$)

$$f \to \int_0^1 f(t)\,dt \tag{3.1}$$

clearly annihilates $[t^k - (k+2)/(k+1)t^{k+1}\colon k \geqslant 0]$. One might be led to conjecture that if the $\alpha_k \to c$ with $1/c \in [a,b]$ then surely the set $[t^k - \alpha_k t^{k+1}\colon k \geqslant 0]$ would not be dense. The next example shows this to be false.

Example 3. $\{\alpha_k = k/(k+1)\}$ on $C[0,2]$. Just as in Example 2 we have $\alpha_k \to 1 \in [0,2]$, we will see that $[t^k - k/(k+1)t^{k+1}\colon k \geqslant 0] = C[0,2]$. Note that the linear functional δ_1'

$$p \to p'(1) \tag{3.2}$$

annihilates S_α and has no continuous extension to $C[0,2]$. Thus it follows that S_α must be dense in $C[0,2]$.

4. The general moment problem

Let M be the set of all sequences of real numbers indexed from 0 to ∞. Given such a sequence, $\{\beta_k\}_{k=0}^{\infty}$, the general moment problem is to determine necessary and sufficient conditions for there to exist a measure $d\mu$ so that for every k

$$\beta_k = \int_a^b t^k \, d\mu.$$

There are many variations on this problem and the standard treatise is the monograph by Ahkiezer [1]. The variations depend on the interval $[a,b]$, the measure $d\mu$ and on the sequence $\{\beta_k\}_{k=0}^{\infty}$. A commonality to the entire set of moment problems is that there are few necessary and sufficient conditions in the literature. It is well known that (for finite intervals $[a,b]$) the topological dual, $C[a,b]^*$, of $C[a,b]$ can be identified with the set of measures of finite variation on $[a,b]$. Thus for a fixed interval $[a,b]$ the existence of the solution to the moment problem above is equivalent to the existence of an $F \in C[a,b]^*$ satisfying

$$\beta_k = F(t^k) \quad \text{for } k = 0, 1, \ldots.$$

We first begin with a theorem that links the moment problem to a density question.

Theorem 4.1. *Fix $[a,b]$. The moment problem*

$$\beta_k = F(t^k) \quad \textit{for } k = 0, 1, \ldots \tag{4.1}$$

has a solution $F \in C[a,b]^*$ if and only if

$$[\beta_k t^j - \beta_j t^k\colon\ 0 \leqslant k, j] \neq C[a,b]. \tag{4.2}$$

Proof. Let $\beta = \{\beta_k\}_{k=0}^{\infty}$. We note that the case $\beta = \theta$, where θ is the zero sequence, is trivial and leave it to the reader. Now assume that $\beta \neq \theta$. If we have a solution to the moment problem as postulated above, then there exists a nontrivial linear functional $F \in C[a,b]^*$ that annihilates the subspace $[\beta_k t^j - \beta_j t^k\colon\ 0 \leqslant k, j] \neq C[a,b]$. Hence this subspace is not $C[a,b]$.

Conversely, with $\beta \neq \theta$ and $[\beta_k t^j - \beta_j t^k\colon\ 0 \leqslant k, j] \neq C[a,b]$ there exists by the Hahn–Banach theorem a nontrivial $F \in C[a,b]^*$ that annihilates the subspace $[\beta_k t^j - \beta_j t^k\colon\ 0 \leqslant k, j]$. The density of the polynomials in $C[a,b]$ guarantees that there is a nonnegative integer m so that $F(t^m) \neq 0$. Fixing m we have

$$F(\beta_m t^j - \beta_j t^m) = 0 \quad \text{for } j = 0, 1, \ldots. \tag{4.3}$$

Since $\beta \neq \theta$ there is an index j_0 so that $\beta_{j_0} \neq 0$. For this index we have

$$\beta_m F(t^{j_0}) = \beta_{j_0} F(t^m). \tag{4.4}$$

Since the right hand side does not vanish we conclude that $\beta_m \neq 0$.

Thus we may define a new functional $G \in C[a,b]^*$ by

$$G = \frac{\beta_m}{F(t^m)} F. \tag{4.5}$$

Clearly $G(t^m) = \beta_m \neq 0$ and $G \perp [\beta_k t^j - \beta_j t^k\colon\ 0 \leqslant k, j]$. It now follows from Eq. (4.3) that $G(t^j) = \beta_j$ completing the proof. $\square$

We will now begin tying this result back to the discussion in the previous section. The following lemma will be useful in the sequel.

Lemma 4.2. *If $\beta_k \neq 0$ for $k = 0, 1, \ldots,$ then*

$$\operatorname{span}\{\beta_k t^j - \beta_j t^k\colon\ 0 \leqslant k, j\} = \operatorname{span}\{\beta_{k+1} t^k - \beta_k t^{k+1}\colon\ 0 \leqslant k\}. \tag{4.6}$$

Proof. Fix p a nonnegative integer. Then it is easy to see by induction that for any nonnegative integer j

$$\beta_{p+j} t^p - \beta_p t^{p+j} \in \operatorname{span}\{\beta_{k+1} t^k - \beta_k t^{k+1}\colon\ 0 \leqslant k\}. \tag{4.7}$$

Clearly the result is true for $j = 0, 1$. Suppose that it is true for $j \leqslant m$ and consider

$$\begin{aligned} &\beta_{p+m+1}(\beta_{p+m} t^p - \beta_p t^{p+m}) + \beta_p(\beta_{p+m+1} t^{m+p} - \beta_{p+m} t^{p+m+1}) \\ &\quad = \beta_{p+m}(\beta_{p+m+1} t^p - \beta_p t^{p+m+1}). \end{aligned} \tag{4.8}$$

Since $\beta_{p+m} \neq 0$ this completes the proof. $\square$

Combining Lemma 4.2 with Theorem 4.1 we have

Corollary 4.3. *If $\beta_k \neq 0$ for $k = 0, 1, \ldots,$ then*

$$\beta_k = F(t^k) \quad \text{for } k = 0, 1, \ldots \tag{4.9}$$

has a solution $F \in C[a, b]^*$ if and only if

$$[\beta_{k+1} t^k - \beta_k t^{k+1}\colon\ 0 \leqslant k] \neq C[a, b]. \tag{4.10}$$

Furthermore, in the case where no moment (i.e., β_k) vanishes we see that

$$\begin{aligned} &\operatorname{span}\{\beta_{k+1} t^k - \beta_k t^{k+1}\colon\ 0 \leqslant k\} = \operatorname{span}\{t^k - \alpha_k t^{k+1}\colon\ 0 \leqslant k\} \\ &\quad \text{where } \alpha_k = \frac{\beta_k}{\beta_{k+1}}. \end{aligned} \tag{4.11}$$

Thus we see that the density of $[t^k - \alpha_k t^{k+1}\colon\ 0 \leqslant k]$ is equivalent to the solvability of a specific moment problem provided that none of the α_k vanish. We record this in the following corollary.

Corollary 4.4. *If $\alpha_k \neq 0$ for $k = 0, 1, \ldots,$ then*

$$\frac{1}{\prod_{i=0}^{k-1} \alpha_i} = F(t^k) \quad \text{for } k = 0, 1, \ldots \tag{4.12}$$

has a solution $F \in C[a, b]^*$ if and only if

$$[t^k - \alpha_k t^{k+1}\colon\ 0 \leqslant k] \neq C[a, b]. \tag{4.13}$$

Continuing this analysis (when none of the β's vanish) we can make a few more observations. First, since the set of β for which the moment problem is solvable is a subspace then we can see that the set of α's that are of one sign are arc-wise connected in the space of sequences. Second, since the moment problem is not solvable for all moments, we conclude that the set of moments does not contain an open set in the space of sequences endowed with the topology of convergence on compact (i.e., finite) subsets.

Let us now consider a simple case where we have one string of zeros in the sequence β. Specifically let us assume that

$$\beta = (\beta_1, \beta_2, \ldots) \quad \text{with } \beta_{k+1} = \cdots = \beta_{k+r} = 0 \text{ and } \beta_j \neq 0 \text{ otherwise.} \tag{4.14}$$

It is easy to see that

$$\begin{aligned}
&\operatorname{span}\{\beta_k t^j - \beta_j t^k \colon 0 \leqslant k, j\} \\
&\quad = \operatorname{span}\bigl\{t^j \colon j \in [k+1, k+r]\bigr\} \\
&\qquad + \operatorname{span}\bigl\{\beta_j t^m - \beta_m t^j \colon j, m \in [0, k] \cup [k+r+1, \infty)\bigr\} \\
&\quad = \operatorname{span}\bigl\{t^j \colon j \in [k+1, k+r]\bigr\} + \operatorname{span}\{\beta_{j+1} t^j - \beta_j t^{j+1} \colon j < k \text{ or } j > k+r\} \\
&\qquad + \operatorname{span}\{\beta_k t^{k+r+1} - \beta_{k+r+1} t^k\}.
\end{aligned} \tag{4.15}$$

Notice that we can take consecutive linear combinations if a consecutive string of β_i's are not zero and we must then augment these functions with the monomials corresponding to the zero β_i's and with the "bridge functions" to leap the gap caused by the zeros.

Let us consider a more complex case with an infinite number of β_i's being zero. In particular, consider the sequence

$$\beta_{2k+1} = 1, \quad \beta_{2k} = 0, \quad k = 0, 1, \ldots. \tag{4.16}$$

It is easy to see that

$$\operatorname{span}\{\beta_k t^j - \beta_j t^k \colon 0 \leqslant k, j\} = \operatorname{span}\{t^{2k+1} - t^{2k+3} \colon 0 \leqslant k\} + \operatorname{span}\{t^{2k} \colon 0 \leqslant k\}. \tag{4.17}$$

Since $[t^{2k} \colon k \geqslant 0] = C[0,1]$ we conclude that the moment problem specified in (4.16) is not solvable on this interval. However, it is easy to see that the same moment problem is solvable relative to the interval $[-1,1]$ since

$$F(f) := \frac{f(1) - f(-1)}{2} \tag{4.18}$$

produces an $F \in C[-1,1]^*$ that annihilates $[t^{2k}, t^{2k+1} - t^{2k+3} \colon k \geqslant 0]$.

Finally notice that for any $[a,b]$ a proper subset of $[-1,1]$ this same moment problem is not solvable since it is easy to see that $[t^{2k}, t^{2k+1} - t^{2k+3} \colon k \geqslant 0] = C[a,b]$.

5. L^2 moment problem

Let M be the set of all sequences of real numbers indexed from 0 to ∞. Given such a sequence, $\{\beta_k\}_{k=0}^{\infty}$, the general $L^2[a,b]$ moment problem is to determine necessary and sufficient conditions for there to exist a $g \in L^2[a,b]$ so that for every k

$$\beta_k = \int_a^b t^k g(t)\,dt.$$

Consider the finite moment problem: find a $g_n \in L^2[a,b]$ that satisfies

$$\beta_k = \int_a^b t^k g_n(t)\,dt \quad \text{for } k \leqslant n. \tag{5.1}$$

Now it is known that there is a unique element of minimal norm g_n^* that satisfies the Eq. (5.1). Furthermore [4] this element g_n^* must be a polynomial of degree less than of equal to n. Thus we have for $k \leqslant n$

$$\begin{aligned}\beta_k &= \int_a^b t^k g_n(t)\,dt = \int_a^b t^k g_n^*(t)\,dt = \int_a^b t^k \sum_{j=0}^{n} c_j^n t^j\,dt = \sum_{j=0}^{n} c_j^n \int_a^b t^k t^j\,dt \\ &= \sum_{j=0}^{n} c_j^n \big[b^{k+j+1} - a^{k+j+1}\big]/(k+j+1) = (H_n \mathbf{c}^n)(k).\end{aligned} \tag{5.2}$$

The matrix

$$H_n(i,j) := \int_a^b t^i t^j\,dt, \quad 0 \leqslant i, j,$$

is the classical Hilbert matrix $H_n(i,j) = 1/(i+j+1)$ if $[a,b] = [0,1]$.

Now the moment problem is solvable in $L^2[a,b]$ if and only if the sequence of functions g_n^* satisfies

$$\sup_{n \geqslant 0} \int_a^b \left(g_n^*\right)^2 dt < \infty.$$

This condition can be written in a more revealing form by setting $\boldsymbol{\beta}^n = (\beta_0, \ldots, \beta_n)'$:

$$\int_a^b \left(g_n^*\right)^2 dt = \int_a^b \left(\sum_{j=0}^{n} c_j^n t^j\right)^2 dt = \boldsymbol{\beta}^{n\prime} H_n^{-1} H_n H_n^{-1} \boldsymbol{\beta}^n = \boldsymbol{\beta}^{n\prime} H_n^{-1} \boldsymbol{\beta}^n. \tag{5.3}$$

We are thus led to the following conclusion.

Theorem 5.1. *Fix* $[a,b]$. *The moment problem*

$$\beta_k = \int_a^b t^k g(t)\,dt \quad \text{for } k = 0, 1, \ldots \tag{5.4}$$

has a solution $g \in L^2[a,b]$ *if and only if*

$$\left[\beta_k t^j - \beta_j t^k \colon 0 \leqslant k, j\right]_{L^2[a,b]} \neq L^2[a,b]. \tag{5.5}$$

Equivalently, the monotone sequence of numbers

$$\boldsymbol{\beta}^{n\prime} H_n^{-1} \boldsymbol{\beta}^n \tag{5.6}$$

is bounded, where H_n *is the* n*th Hilbert matrix with respect to the interval* $[a,b]$.

Since for finite intervals $L^2[a,b] \subset L^1[a,b]$, we see that non-density in $L^2[a,b]$ implies non-density in $C[a,b]$. We record this result in the following corollary.

Corollary 5.2. *If the sequence in* (5.6) *is bounded then*

$$\left[\beta_k t^j - \beta_j t^k \colon 0 \leqslant k, j\right] \neq C[a,b].$$

There is one setting where we can explicitly write down simple necessary and sufficient conditions for density of binomial families and it bears on density in $L^2[-1,1]$ as well. The Hardy space H^2 is the usual subspace of holomorphic functions f on the open unit disc satisfying

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{with } \sum_{n=0}^{\infty} |a_n|^2 < \infty. \tag{5.7}$$

It is clear that these functions have an L^2 trace on the boundary of the disc so it is common to write

$$\|f\|^2 := \frac{1}{2\pi} \int_0^{2\pi} \left|f(e^{i\theta})\right|^2 d\theta.$$

Now consider the set of polynomials in H^2

$$S_\alpha = \operatorname{span}\{z^n - \alpha_n z^{n+1} \colon n = 0, 1, \ldots\}.$$

Let us assume that $\alpha_i \neq 0$ for all i. Then we can prove the following simply stated result.

Theorem 5.3. *The set* S_α *is dense in* H^2 *if and only if the sequence*

$$\left\{ \frac{1}{\prod_{i=0}^{n-1} \alpha_i} \right\} \notin \ell^2. \tag{5.8}$$

The proof is straightforward. If the subspace S_α is not dense then there exists an annihilating function $g \neq 0$. That means that g satisfies

$$0 = \frac{1}{2\pi}\int_0^{2\pi} \left(e^{in\theta} - \alpha_n e^{i(n+1)\theta}\right)\bar{g}\left(e^{i\theta}\right) d\theta, \quad n = 0, \ldots.$$

Thus in terms of the Fourier expansion of g we have

$$\hat{g}(n) - \bar{\alpha}_n \hat{g}(n+1) = 0, \quad n = 0, \ldots.$$

Solving this difference equation (and recalling that $g \neq 0$) yields (5.8).

Now using the Riesz–Fejer theorem, [2], we obtain an easily verifiable sufficient result for density in $L^2[-1,1]$.

Corollary 5.4. *Assuming that the $\alpha_n \neq 0$ for all n then*

$$[t^n - \alpha_n t^{n+1}\colon n \geqslant 0]_{L^2[-1,1]} = L^2[-1,1] \tag{5.9}$$

if the sequence

$$\left\{\frac{1}{\prod_{i=0}^{n-1}\alpha_i}\right\} \notin \ell^2. \tag{5.10}$$

The proof of this result is straightforward since if we can approximate $f(z) = 1$ in H^2, then we can approximate it in $L^2[-1,1]$ by the Riesz–Fejer theorem. But once

$$1 \in [t^n - \alpha_n t^{n+1}\colon n \geqslant 0]_{L^2[-1,1]}$$

it is easy to see that the subspace is dense since it contains all polynomials (by solving forward). The converse of this theorem is not true since the sequence $\alpha_n = 2$ for all n satisfies

$$\left\{\frac{1}{\prod_{i=0}^{n-1}\alpha_i}\right\} \in \ell^2, \quad \text{but} \quad [t^n - 2t^{n+1}\colon n \geqslant 0]_{L^2[-1,1]} = L^2[-1,1]. \tag{5.11}$$

It is interesting to note that as a result of the above observation we can make the following statement: restricting our attention to $L^2[-1,1]$ and

$$H(i,j) = \int_{-1}^{1} t^i t^j\, dt = \frac{1-(-1)^{i+j+1}}{i+j+1}, \quad 0 \leqslant i, j,$$

if $\beta \notin \ell^2$ then

$$(\beta^n)' H_n^{-1} \beta^n \to \infty.$$

We are not aware of a direct proof of this fact.

6. Operator theoretic equivalences

Let $\{\alpha_k\}$ be any sequences of real numbers. The primary question of this paper is when is the span of the set $S_\alpha = \{t^n - \alpha_n t^{n+1}\colon\ n = 0, 1, \ldots\}$ dense in $C[a, b]$. We will show in this section that this question is closely related to a question of determining the range of a certain operator that is in general at most densely defined in $C[a, b]$.

Definition 6.1. For every n define $T_\alpha(t^n) = \alpha_n t^n$. Extend T linearly to an operator defined on the set of polynomials, $P[a, b]$, defined as a subset of $C[a, b]$.

Definition 6.2. Let $G_\alpha = I - tT_\alpha$ be defined as an operator on $P[a, b]$ as

$$G_\alpha\bigl(p(t)\bigr) = p(t) - tT_\alpha\bigl(p(t)\bigr).$$

It is clear that the span $S_\alpha = \operatorname{range} G_\alpha$. Thus it suffices to classify the closure of the range of G_α in order to classify the closure of the span of S_α. We note that if G_α is continuous on $P[a, b]$ then it can be extended to $C[a, b]$ and to prove that S_α is dense it would suffice to show that the equation

$$G_\alpha(x) = 1$$

has a solution. We begin with a negative result that shows that the operator is not always continuous even for “nice” sequences.

Example 1. Consider the sequence

$$\alpha_n = \frac{(-1)^n}{n+1}.$$

Now the sequence of polynomials

$$p_n(t) = (1-t)^n$$

is a bounded sequence in the space $C[0, 1]$. We will show that $G_\alpha(p_n(t))$ is unbounded and hence that G_α is discontinuous. Write

$$(1-t)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i t^i.$$

Now calculate

$$tT_\alpha\bigl((1-t)^n\bigr) = \sum_{i=0}^{n} \binom{n}{i} (-1)^i \frac{(-1)^i}{i+1} t^{i+1} = \sum_{i=0}^{n} \binom{n}{i} \frac{1}{i+1} t^{i+1}.$$

Now consider the quantity

$$\begin{aligned}\binom{n}{i}\frac{1}{i+1} &= \frac{n!}{i!(n-i)!}\frac{1}{i+1} = \frac{n!}{(i+1)!(n-i)!}\\ &= \frac{1}{n+1}\frac{(n+1)!}{(i+1)!(n-i)!} = \frac{1}{n+1}\binom{n+1}{i+1}.\end{aligned}$$

We now continue the calculation:

$$
\begin{aligned}
tT_\alpha\big((1-t)^n\big) &= \sum_{i=0}^{n} \binom{n}{i} \frac{1}{i+1} t^{i+1} = \sum_{i=0}^{n} \frac{1}{n+1} \binom{n+1}{i+1} t^{i+1} \\
&= \frac{1}{n+1}(t+1)^{n+1} - \frac{1}{n+1}.
\end{aligned}
$$

We now have that

$$
G_\alpha\big((1-t)^n\big) = (1-t)^n - \frac{1}{n+1}(t+1)^{n+1} + \frac{1}{n+1}
$$

and we see by evaluating at $t = 1$ that the norm of

$$
\|G_\alpha\| \geqslant \frac{2^{n+1} - 1}{n+1}
$$

and hence G_α is unbounded. On the other hand, we know that that the image of G_α is dense despite the fact that G_α is not continuous. In fact though, G_α is densely defined on $C[a, b]$.

Example 2. Consider the sequence $\alpha_n = n + 2$ and define T_α as before. Now

$$
tT_\alpha(t^n) = (n+2)t^{n+1} = \frac{d}{dt} t^{n+2}
$$

and hence

$$
G_\alpha\big(p(t)\big) = p(t) - D\big(t^2 p(t)\big),
$$

and we ask if we can solve the equation

$$
p(t) - D\big(t^2 p(t)\big) = 1?
$$

It is clear that for intervals $0 < a < b$ that we have local solutions to the differential equation

$$
\dot{f} = \frac{(1-2t)f - 1}{t^2}.
$$

If there are such solutions then it is clear that these solutions and their derivatives can be uniformly approximated by polynomials on $[a, b]$ and hence we obtain a sequence of polynomials p_n having the property that

$$
G_\alpha(p_n) \to 1
$$

in $C[a, b]$.

Example 3. Now consider the sequence $\alpha_n = 1/(n+1)$ and define T_α as before. Now

$$
tT_\alpha(t^n) = \frac{1}{n+1} t^{n+1} = \int_0^t s^n \, ds
$$

and hence

$$G_\alpha\big(p(t)\big) = p(t) - \int_0^t p(s)\,ds$$

is continuous and can be extended to $C[a, b]$. Now we consider the equation

$$f(t) - \int_0^t f(s)\,ds = 1$$

and we see that

$$f(t) = e^t$$

is a solution. Thus 1 is in the image of G_α and hence G_α is surjective. We, of course, know this from earlier results.

Example 4. Consider the sequence $\alpha_n = n + 1$ and define T_α as before. We now work in the space $C[0, 1]$. Now

$$tT_\alpha(t^n) = (n+1)t^{n+1} = tD(tt^n)$$

and hence

$$G_\alpha\big(p(t)\big) = p(t) - tD\big(tp(t)\big).$$

This operator is not continuous but is densely defined. We can ask if the differential equation

$$p - tD(tp) = 1$$

has a continuous solution. Rewriting the equation we have

$$t^2\dot{p} = (1-t)p - 1.$$

The solution of the homogeneous equation is

$$p(t) = ke^{-1/t}\frac{1}{t}$$

and using variation of parameters we have

$$k(t) = -\int_t^1 \frac{e^{1/t}}{t}\,dt,$$

and thus a solution of the differential equation is given by

$$p(t) = -\frac{e^{-1/t}}{t}\int_t^1 \frac{e^{1/s}}{s}\,ds.$$

It remains to decide if the solution is continuous at $t = 0$. Rewriting the solution as

$$p(t) = \frac{-\int_t^1 \frac{e^{1/s}}{s}\,ds}{te^{1/t}}$$

and taking the limit we have that

$$\lim_{t\to 0} p(t) = -1$$

and hence the solution can be continuously defined on the interval $[0, 1]$.

We can conclude that the span of the set

$$S_\alpha = \left\{t^n - (n+1)t^{n+1}\colon n = 0, 1, \ldots\right\}$$

is dense in $C[0, 1]$ and as a consequence the sequence $m_k = m_0/(k + 1)!$ is not a sequence of moments for any function of bounded variation on the interval $[0, 1]$.

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